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# Statistics of anomalously localized states at the center of band $E = 0$ in the one-dimensional Anderson localization model

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**Abstract.** We consider the distribution function  $P(|\psi|^2)$  of the eigenfunction amplitude at the center-of-band ( $E = 0$ ) anomaly in the one-dimensional tight-binding chain with weak uncorrelated on-site disorder (the one-dimensional Anderson model). The special emphasis is on the probability of the anomalously localized states (ALS) with  $|\psi|^2$  much larger than the inverse typical localization length  $\ell_0$ . Using the solution to the generating function  $\Phi_{an}(u, \phi)$  found recently in our works [18, 17] we find the ALS probability distribution  $P(|\psi|^2)$  at  $|\psi|^2 \ell_0 \gg 1$ . As an auxiliary preliminary step we found the asymptotic form of the generating function  $\Phi_{an}(u, \phi)$  at  $u \gg 1$  which can be used to compute other statistical properties at the center-of-band anomaly. We show that at moderately large values of  $|\psi|^2 \ell_0$ , the probability of ALS at  $E = 0$  is smaller than at energies away from the anomaly. However, at very large values of  $|\psi|^2 \ell_0$ , the tendency is inverted: it is exponentially easier to create a very strongly localized state at  $E = 0$  than at energies away from the anomaly. We also found the leading term in the behavior of  $P(|\psi|^2)$  at small  $|\psi|^2 \ll \ell_0^{-1}$  and show that it is consistent with the exponential localization corresponding to the Lyapunov exponent found earlier by Kappus and Wegner [8] and Derrida and Gardner [9].

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## 1. Introduction

There is a long-lasting interest in localization effects [1, 2] in 1d systems [3]-[18]. The simplest and most widely studied model is a linear chain with a nearest-neighbor hopping and random site energies  $\varepsilon_i$  with no inter-site correlation:  $\langle \varepsilon_i \varepsilon_j \rangle = \delta_{ij} \sigma^2$ . The wave function  $\psi_i$  at a site  $i$  of this one-dimensional Anderson localization model [1] obeys the equation:

$$\psi_{i-1} + \psi_{i+1} + \varepsilon_i \psi_i = E \psi_i. \quad (1)$$

In the absence of disorder ( $\varepsilon_i \equiv 0$ ) the eigenstates would be plane waves, with eigenenergies determined by the wave vector  $k$ :  $E(k) = 2 \cos(k)$ ,  $k \in (-\pi, \pi)$ . In the

presence of the disorder, the eigenstates are random and require statistical description. Moreover, the states are *localized* at an arbitrary small disorder strength  $\sigma$ . For weak disorder the localization length  $\ell(E)$  is large as compared to the lattice constant:  $\ell(E) \gg 1$ . This means that the “typical” magnitude of the normalized wave function near its localization center can be estimated as  $|\psi|_{typ}^2 \sim 1/\ell(E) \ll 1$ . However, for some realizations of the disorder, more strongly localized states, “anomalously localized states” (ALS), are possible, with the value of the wave function maximum in the range of  $1/\ell(E) \ll |\psi|^2 \leq 1$  (the right equality would correspond to a state localized at a single lattice site). Our aim in the present paper is to study the probability distribution  $P(|\psi|^2)$  of such strongly localized states in a long *weakly disordered* chain.

We will be especially interested in the statistics of ALS in the vicinity of the so-called Kappus-Wegner center-of-band ( $E = 0$ ,  $k = \pi/2$ ) anomaly [8]. This anomaly is a feature of a discrete chain (it is absent in the continuum model) and originates from the commensurability of the de Broglie wavelength and the lattice constant. The anomaly manifests itself [8, 9] in a sharp, *finite* in the limit  $\sigma \rightarrow 0$ , enhancement of the density of states (DoS)  $\nu(E = 0)$  and the localization length  $\ell(E = 0)$  inside a very narrow energy window (of the width  $\sim \sigma^2$ ) around the band center  $E = 0$  as compared to their values

$$\nu_0(E = 0) \approx \frac{1}{2\pi} ; \quad \ell_0 \equiv \ell_0(E = 0) = \frac{8}{\sigma^2} \quad (2)$$

beyond this interval [19]. In particular, it was shown [9] that in the limit  $\sigma \ll 1$ :

$$\frac{\nu(E = 0)}{\nu_0(E \rightarrow 0)} = \frac{4\sqrt{2}\pi^3}{\Gamma^4(1/4)} = 1.01508... ; \quad \frac{\ell^{ext}(E = 0)}{\ell_0(E = 0)} = \frac{1}{16\pi^2} \Gamma^4\left(\frac{1}{4}\right) = 1.0942... \quad (3)$$

Here we have introduced the superscript “*ext*” to emphasize that the corresponding localization length  $\ell^{ext} = 1/[\Re \gamma(E)]$  is defined by the Lyapunov exponent  $\gamma(E)$  and therefore characterizes the exponentially decaying *tails* of localized wave functions; for this reason it will be referred to as the *extrinsic* localization length. Similar anomalies have been found later [15, 16] for other physical quantities (like transmission and conductance), also related with the Lyapunov exponent.

In contrast to this set of problems, the eigenfunction statistics  $P(|\psi|^2)$  may provide information about an “intrinsic” spatial structure of localized wave functions including the vicinity of the center of localization. In particular, it allows to calculate the “intrinsic” localization length  $\ell^{int}(E) = 1/I_2(E)$ , where  $I_2(E) = \sum_i |\psi_i(E)|^4$  is the inverse participation ratio.

However, studying the statistical properties of *normalized* eigenfunctions is a considerably more difficult theoretical problem than studying the Lyapunov exponent (the latter is related to propagation of an external wave in a semi-infinite chain and is not directly related with *eigenfunctions*).

The formalism for studying the eigenfunction statistics in a disordered chain (see review [20]), adapted recently [18] to the case of the center-of-band anomaly, expresses moments of the eigenfunction distribution in terms of a “generating function”  $\Phi(u, \phi; E)$  of the two auxiliary variables. These variables can be loosely interpreted [8, 18] as the squared amplitude  $u \sim |a_j|^2 \ell_0$  and the “phase”  $\phi$  defined by a representation of

eigenfunctions in the form:  $\psi_j = a_j \cos(kj + \phi_j)$  with slowly varying  $a_j > 0$  and  $\phi_j \in (0, \pi)$ . The generating function  $\Phi(u, \phi; E)$  allows one to calculate all local statistics of eigenfunctions. In particular, it determines the inverse participation ratio (IPR)  $I_2$  and higher moments  $I_m = \sum_j \langle |\psi_j|^{2m} \rangle$ , as well as the full distribution function  $P(|\psi|^2)$ . Also, the generating function  $\Phi(u, \phi; E)$  determines (through a *nonlinear* integral relation Eq.(60)) the joint probability distribution  $P(u, \phi; E)$  of the amplitude and the phase. However, the relationship between the generating function  $\Phi(u, \phi; E)$  and the normalized distribution function  $\mathcal{P}(\phi; E) = \int du P(u, \phi; E)$  of the phase  $\phi$  turns out to be remarkably simple [18], it is given by the limit  $u \rightarrow 0$  of the generating function  $\Phi(u, \phi; E)$ :

$$\Phi(u = 0, \phi; E) = \mathcal{P}(\phi; E) = 2P_{refl}(\theta; E)|_{\theta=2\phi}, \quad (4)$$

There is also a simple relationship between  $\mathcal{P}(\phi; E)$  and the probability distribution  $P_{refl}(\theta; E)$  of the reflection phase  $\theta$  for a wave incident on a semi-infinite disordered chain. It is given by the second equality in Eq.(4). At weak disorder the phase distribution  $\mathcal{P}(\phi; E)$  is uniform in the continuum model and outside the center-of-band anomaly but it becomes a non-trivial function of  $\phi$  at  $E = 0$  [9].

A relative simplicity of calculation of such quantities as the Lyaupunov exponent (and the extrinsic localization length  $\ell^{\text{ext}}(E)$ ) and the DoS,  $\nu(E)$ , is due to the fact that they can be expressed entirely in terms of the the probability distribution  $\mathcal{P}(\phi; E)$ , i.e. involve the generating function  $\Phi(u = 0, \phi; E)$  at  $u = 0$ . For instance, the DoS,  $\nu(E)$  is given by [8, 18]:

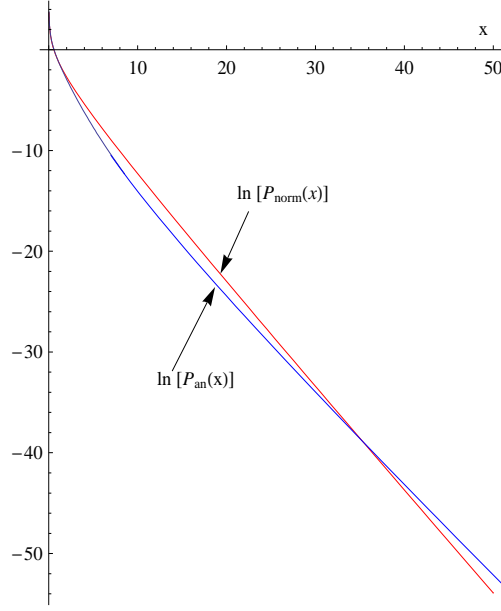
$$\frac{\nu(E)}{\nu_0(E)} = 4\pi \int_0^{\pi/2} d\phi \cos^2(\phi) [\mathcal{P}(\phi; E)]^2. \quad (5)$$

On the contrary, the complexity of the problem of local eigenfunction statistics arises because it requires the full generating function  $\Phi(u, \phi; E)$  of the two variables  $u$  and  $\phi$ . In particular, the statistics of relatively rare anomalously localized eigenstates of large peak amplitude  $|\psi|^2 \ell_0 \gg 1$  which we will study in the present paper is determined by  $\Phi(u, \phi; E)$  at large values of the variable  $u \gg 1$ .

The generating function  $\Phi(u, \phi; E)$  for a long chain at the center-of-band anomaly has been found recently [17, 18] by solving exactly the corresponding second order partial differential equation Eq.(10) in  $u$  and  $\phi$  variables. The exact solution Eq.(14) to this equation reflects a hidden symmetry of the problem which has not been yet explicitly exploited. However, the solution is given in quadratures as an integral of a product of Whittaker functions over the variable which enters both the argument and the index of these functions. In this paper we perform a careful analysis of the integral and derive the asymptotic form of  $\Phi(u, \phi)$  at large values of  $u \gg 1$  (from now on we omit the energy argument  $E = 0$  for brevity). It has a form:

$$\Phi_{an}(u, \phi) = A(\phi) \frac{e^{-\sqrt{u}b(\phi)}}{u^{1/4}} \quad ; \quad u \gg 1, \quad (6)$$

where the function  $b(\phi)$  is a solution to the first order ordinary differential equation (54), and  $A(\phi)$  is specified in the section 2.



**Figure 1.** The logarithm of the ALS probability distribution  $P(x)$  ( $x = |\psi|^2 \ell_0$ ) at the center-of-band anomaly  $E = 0$  and outside ( $E \neq 0$ ). At moderately large values of  $|\psi|^2 \ell_0$  ( $|\psi|^2 \ell_0 < 35$ ) the probability of ALS at  $E = 0$  is smaller than that for  $E \neq 0$ . However, at  $|\psi|^2 \ell_0 > 35$  the situation is inverted: the probability ( $\sim 10^{-15}$ ) of very strongly localized states is larger at the band center. Note that at weak disorder  $\sigma \ll 1$  we consider in this paper the typical localization length  $\ell_0 \approx 8/\sigma^2 \gg 1$  is parametrically larger than the lattice constant  $a = 1$  [19]. Thus the above results are valid when the anomalously small localization length is still much larger than the lattice constant.

It allows us to compute the tail of the distribution function  $P(|\psi|^2)$  at  $|\psi|^2 \ell_0 \gg 1$  in a long chain of the length  $L \gg \ell_0$ :

$$P_{an}(|\psi|^2) \sim \frac{1}{\ell_0 L} \frac{\exp(-\kappa |\psi|^2 \ell_0)}{|\psi|^6}, \quad (|\psi|^2 \ell_0 \gg 1), \quad (7)$$

where the coefficient  $\kappa$  is determined by some “critical angle”  $\phi_c$ , given by Eqs.(39),(40), at which the function  $b^2(\phi)/4 \cos^2 \phi$  reaches its minimum:

$$\kappa = \frac{b^2(\phi_c)}{4 \cos^2 \phi_c} = 0.830902... < 1. \quad (8)$$

The anomalous distribution of eigenfunction amplitudes Eq.(7) should be compared with the “normal” one [18] valid in the continuum model and outside the center-of-band anomaly in the discrete chain [23]:

$$P_{norm}(|\psi|^2) = \frac{\ell_0}{L} \frac{\exp(-|\psi|^2 \ell_0)}{|\psi|^2}, \quad (|\psi|^2 \ell_0 \gg e^{-L/\ell_0}). \quad (9)$$

A comparison of Eqs.(7) and (9) reveals an unexpected feature (see Fig.1). While the probability of moderately strongly localized states (with the peak intensity  $1 < |\psi|^2 \ell_0 < 35$ ) is smaller at  $E = 0$  than that away from the anomaly, very strongly localized states (with  $|\psi|^2 \ell_0 > 35$ ) are more probable at the band center. Formally this re-entrant

behavior is caused by the value of  $\kappa \approx 0.83 < 1$  (see Eq.(8)) at the "critical angle"  $\phi_c \neq 0$ ; for the "normal" case  $E \neq 0$  one obtains  $b(\phi) = 2$  and thus  $\phi_c = 0$  and  $\kappa = 1$ .

The behavior of moderately strongly localized states is consistent with the result Eq.(3) for the Lyapunov exponent which gives an enhanced typical extrinsic localization length at  $E = 0$ . The opposite trend for very strongly localized states is perhaps due to the Bragg-mirror effect of the harmonics of the random potential which double the period of the lattice [21, 22].

A point of special interest is the distribution of small amplitudes  $P(|\psi|^2)$  at  $|\psi|^2 \ell_0 \ll 1$ , as it gives an idea on the shape of the tail of the localized wave function. We found the leading term  $|\psi|^{-2}$  in  $P(|\psi|^2)$  at small  $|\psi|^2 \ell_0$  and shown that it is universal for all systems with exponentially localized eigenstates.

The rest of the paper is devoted to the derivation of the announced results and is organized in the following way. In section 2 we obtain the asymptotic of the generating function  $\Phi_{an}(u, \phi)$  at  $u \gg 1$ . In the subsequent section 3 we derive the asymptotic of the probability distribution function  $P_{an}(|\psi|^2)$  at  $|\psi|^2 \ell_0 \gg 1$ . The behavior of  $P(|\psi|)$  at small  $|\psi|^2 \ell_0$  is analyzed in Sec.4. In the last section 5 we summarize and discuss the obtained results.

## 2. Generating function $\Phi_{an}(u, \phi)$ and its asymptotic at $u \gg 1$

Sufficiently far from the ends of a long chain, the generating function becomes site independent. At the center-of-band anomaly ( $E = 0$ ) this *stationary* generating function,  $\Phi_{an}(u, \phi)$ , obeys the partial differential equation (PDE) [17, 18]

$$\left[ [1 - \cos(4\phi)] u^2 \partial_u^2 + \sin(4\phi) u \partial_u \partial_\phi + \frac{3 + \cos(4\phi)}{4} \partial_\phi^2 + 2 \cos(4\phi) u \partial_u - \frac{3}{2} \sin(4\phi) \partial_\phi - 2 \cos(4\phi) - u \right] \Phi_{an}(u, \phi) = 0 \quad (10)$$

Its solution should also meet the requirements of being a smooth periodic function of  $\phi$ , regular, positive and non-zero at  $u \rightarrow 0$  (we recall that  $\Phi(u = 0, \phi)$  is the phase distribution function, see Eq.(4)) and decaying at  $u \rightarrow \infty$ .

These requirements are rather restrictive. For instance, the solution

$$\Phi_0(u, \phi) = u \exp \left( -\sqrt{u} (|\cos \phi| + |\sin \phi|) \right) \quad (11)$$

is not appropriate for it is not a smooth function of  $\phi$ .

For comparison, we write down also the equation for the "normal" generating function  $\Phi_{norm}(u, \phi)$  (i.e. when the energy lies outside the anomaly region, or for the continuous model):

$$\left[ u^2 \partial_u^2 - u + \frac{3}{4} \partial_\phi^2 \right] \Phi_{norm}(u, \phi) = 0. \quad (12)$$

This equation looks like a "course-grained" PDE (10) where all the coefficients are "averaged" over the angle interval  $(0, \pi)$  (so called "phase randomization") which is equivalent to course-graining over the space region  $\ell_0 \gg \Delta x \gg 1/k$ . The variables  $u$

and  $\phi$  in Eq.(12) are separated and one immediately finds that the only solution decaying at  $u \rightarrow \infty$  and remaining regular and non-zero at  $u \rightarrow 0$  is given by

$$\Phi_{norm}(u, \phi) = \frac{2}{\pi} \sqrt{u} K_1(2\sqrt{u}) \approx \frac{u^{1/4}}{\sqrt{\pi}} e^{-2\sqrt{u}} \quad \text{at } u \gg 1, \quad (13)$$

where  $K_1(x)$  is the modified Bessel function. This solution has been earlier obtained [13] in the continuous model. It also arises in the theory of a multi-channel disordered wire [24, 20]. The corresponding phase distribution is uniform:  $P_{norm}(\phi) = \Phi_{norm}(u = 0, \phi) = 1/\pi$ .

Unlike Eq.(12), the PDE (10) is not separable in the variables  $u$  and  $\phi$ . However, due to a hidden (and not well established yet) symmetry of the problem, it was possible to find new variables which allowed us to split the PDE (10) into two ordinary differential equation and thus to construct an exact general solution [17, 18]. The solution, which obeys the above requirements, is given by [18]:

$$\Phi_{an}(u, \phi) = \frac{u^{1/2}}{2\Gamma^4\left(\frac{1}{4}\right) |\cos \phi \sin \phi|^{1/2}} \int_0^\infty d\lambda \frac{|\Gamma\left(\frac{1}{4} + \epsilon\lambda\right)|^2}{\lambda^{3/2}} \left[ W_{-\lambda\epsilon, \frac{1}{4}}\left(\frac{\bar{\epsilon} u \cos^2 \phi}{4\lambda}\right) W_{-\lambda\bar{\epsilon}, \frac{1}{4}}\left(\frac{\epsilon u \sin^2 \phi}{4\lambda}\right) + c.c. \right], \quad (14)$$

where  $\epsilon = e^{i\pi/4}$ ,  $\bar{\epsilon} = e^{-i\pi/4}$  and  $W_{\lambda, \mu}(z)$  is the Whittaker function (For the second index  $\mu = 1/4$  the Whittaker function can be expressed also in terms of the parabolic cylinder function, see, e.g. [25]). In the limit  $u \rightarrow 0$  the expression (14) reproduces the phase distribution function  $P_{an}(\phi) = \Phi_{an}(u = 0, \phi)$ :

$$\mathcal{P}_{an}(\phi) = \frac{4\sqrt{\pi}}{\Gamma^2(\frac{1}{4})} \frac{1}{\sqrt{3 + \cos(4\phi)}}, \quad (15)$$

which was derived earlier [9] in a different way. It shows that the phase distribution becomes non-uniform at the center-of-band anomaly.

Our current task is to derive an asymptotic expression for  $\Phi_{an}(u, \phi)$  in the limit of large  $u \gg 1$ . The integrand in Eq.(14) is too complicated for a brute force attack. This is because both the arguments and the first indices of the Whittaker functions are large (as is shown below, the leading contribution to the integral comes from  $\lambda \sim \sqrt{u}$ ) and the standard [25] asymptotic expansions of these functions are not applicable. Our approach will include three steps: first we will represent Eq.(14) in the form which allows us to find an asymptotic expression of the integrand; then we obtain the asymptotic form Eq.(7) of the generating function  $\Phi(u, \phi)$  at large  $u$  (this asymptotic expression will be obtained in the next subsection), and finally the ALS distribution function  $P(|\psi|^2)$  will be found by a saddle-point integration over  $\phi$ .

The generating function Eq.(14) is periodic in  $\phi$  (with the period  $\pi/2$ ) and symmetric with respect to the change  $\phi \rightarrow \pi/2 - \phi$ . Therefore, it is sufficient to calculate  $\Phi(u, \phi)$  in the angular interval  $(0, \pi/4]$ . We exploit the following integral representation

of the Whittaker function (cf. 9.222.1 [25]):

$$W_{-\lambda, \frac{1}{4}}(x) = \frac{\sqrt{2} x^{1/4}}{\Gamma(1/4 + \lambda)} \int_1^\infty e^{-xt/2} \left( \frac{t-1}{t+1} \right)^\lambda \frac{dt}{(t^2-1)^{3/4}} \quad (16)$$

valid for  $\Re x \geq 0$  and  $\Re \lambda \geq 0$ . Since the integrand in Eq.(14) is an analytical function within the sector  $\pi/4 \leq \arg \lambda \leq \pi/4$ , we rotate the integration contour  $\lambda \rightarrow \lambda e^{i\pi/4}$  and introduce a new integration variable  $z$ :

$$\lambda = \frac{1}{4} \sqrt{\frac{u}{z}}. \quad (17)$$

After these transformations, Eq.(14) takes the form:

$$\Phi(u, \phi) = \frac{2\sqrt{u}}{\Gamma^4(1/4)} \Re \int_0^\infty \frac{e^{i\pi/4} dz}{\sqrt{z}} I_1(z, \phi) I_2(z, \phi). \quad (18)$$

Here

$$I_{1(2)}(z, \phi) = \int_1^\infty \frac{dt}{(t^2-1)^{3/4}} \exp[-\sqrt{u} f_{1(2)}(t, z, \phi)], \quad (19)$$

where

$$f_1(t, z, \phi) \equiv \frac{1}{4\sqrt{z}} \ln \left( \frac{t+1}{t-1} \right) + \frac{t\sqrt{z} \cos^2 \phi}{2} \quad (20)$$

is real, while

$$f_2(t, z, \phi) \equiv -\frac{i}{4\sqrt{z}} \ln \left( \frac{t+1}{t-1} \right) + \frac{it\sqrt{z} \sin^2 \phi}{2} \quad (21)$$

is purely imaginary for real  $z$ . Exact Eqs.(18)-(21) constitute the starting point for the calculation of asymptotic expressions at  $u \gg 1$ .

### 2.1. Asymptotic of the integrand in Eq.(18)

At  $u \gg 1$  the integrals Eq.(19) can be computed in the saddle-point approximation. The minimum of the action in the integrand of  $I_1(z, \phi)$  is achieved at the point

$$t_0 = \sqrt{1 + \frac{1}{z \cos^2 \phi}} > 1. \quad (22)$$

The integration contour goes through this point, so the corresponding saddle-point contribution is given by

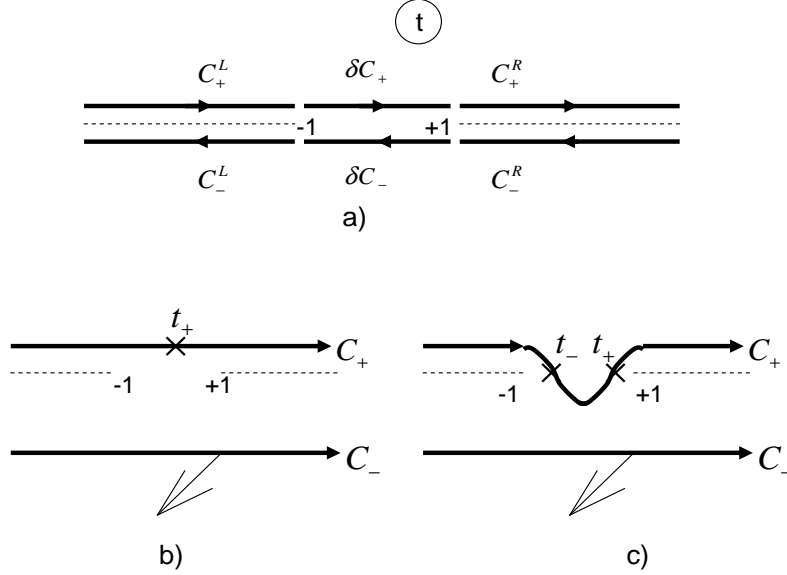
$$I_1^{(s)}(z, \phi) = \frac{\sqrt{2\pi}}{u^{1/4}} \frac{z^{1/4}}{(1 + z \cos^2 \phi)^{1/4}} e^{-\sqrt{u} F_1(z, \phi)}, \quad (23)$$

where

$$F_1(z, \phi) = f_1(t_0, z, \phi) = \frac{\cos \phi}{2} \left[ \frac{\ln \left( \sqrt{1 + z \cos^2 \phi} + \sqrt{z} \cos \phi \right)}{\sqrt{z} \cos \phi} + \sqrt{1 + z \cos^2 \phi} \right]. \quad (24)$$

For the integral  $I_2(z, \phi)$  the situation is more complicated as there are two saddle-points:

$$t_\pm = \pm \begin{cases} i \sqrt{\frac{1}{z \sin^2 \phi} - 1}, & z < 1/\sin^2 \phi, \\ \sqrt{1 - \frac{1}{z \sin^2 \phi}}, & z > 1/\sin^2 \phi \end{cases}, \quad (25)$$



**Figure 2.** Choice of contours (solid lines with arrows) in the complex plane of  $t$ : a) initial contours; b) and c) the final contours deformed to pass through the saddle points. The cuts are denoted by the dotted lines. The contour  $C_-$  can be deformed away to infinity in the lower half-plane.

both lie outside the integration semi-axis  $t > 1$ . On the complex plane  $t$  with two cuts,  $(-\infty, -1)$  and  $(1, \infty)$  we define an integral over a contour  $C$  by

$$I_2[C] \equiv \int_C \frac{dt}{(t^2 - 1)^{3/4}} \exp[-\sqrt{u} f_2(t, z, \phi)], \quad (26)$$

where we choose the branch of the integrand so that on the upper edge of the cut  $(1, \infty)$   $I_2[C_+^R] = I_2(z, \phi)$ . Taking the contour  $C = C_-^L + C_+^L + C_-^R + C_+^R$  as depicted in Fig.2a, one checks straightforwardly that

$$\begin{aligned} I_2[C] &= -2e^{i\pi/4} e^{\pi\sqrt{u}/(2\sqrt{z})} \Re \left( [1 + ie^{-\pi\sqrt{u}/(2\sqrt{z})}] e^{i\pi/4} I_2(z, \phi) \right) \\ &\approx -2e^{i\pi/4} e^{\pi\sqrt{u}/(2\sqrt{z})} \Re \left( e^{i\pi/4} I_2(z, \phi) \right). \end{aligned} \quad (27)$$

Thus, with the exponential accuracy we have expressed the quantity of our interest  $\Re(e^{i\pi/4} I_2(z, \phi))$  (see Eq.(18)) in terms of the contour integral  $I_2[C]$ . Evidently, the latter is not changed if the integration is extended to parts  $\delta C_+$  and  $\delta C_-$  comprising the closed contour (see Fig.2a). In this way we arrive at the important relation:

$$I_2[C] = I_2[C_+] + I_2[C_-] = I_2[C_+]. \quad (28)$$



where the contours  $C_+$  and  $C_-$  are shown in Fig.2b,c; the last equality in Eq.(28) holds because the contour  $C_-$  may be safely shifted down to the infinitely remote part of the half-plane  $\Im t < 0$ , where  $I_2[C_-]$  vanishes (see Eqs.(26) and (21)).

Further transformations depend on the location of the saddle-points, i.e. on the value of  $z$ , see Eqs.(25) and (25). For  $z \sin^2 \phi < 1$ , the contour  $C_+$  can be lifted to the upper half-plane of  $t$  (see Fig.2b) to go through the saddle-point  $t_+$  (25) which provides the minimum of the action. The corresponding saddle-point contribution to  $\Re(e^{i\pi/4} I_2(z, \phi))$  is given by ( $z \sin^2 \phi < 1$ )

$$\Re(e^{i\pi/4} I_2^{(s)}(z, \phi)) = -\frac{e^{-i\pi/4}}{2} e^{-\pi\sqrt{u}/(2\sqrt{z})} I_2^{(s)}[C_+] = \frac{\sqrt{2\pi}}{2u^{1/4}} \frac{z^{1/4} e^{-\sqrt{u} F_2^<(z, \phi)}}{[1 - z \sin^2 \phi]^{1/4}}, \quad (29)$$

where

$$F_2^<(z, \phi) = \frac{\sin \phi}{2} \left[ \frac{1}{\sqrt{z} \sin \phi} \left( \frac{\pi}{2} + \arctan \sqrt{\frac{1}{z \sin^2 \phi} - 1} \right) - \sqrt{1 - z \sin^2 \phi} \right]. \quad (30)$$

When  $z \sin^2 \phi > 1$ , the two saddle points (25) lie on the real axis. We bent the contour  $C_+$  so that it goes through the both points within the proper Stokes sectors (Fig.2c). The resulting saddle-point contribution to  $\Re(e^{i\pi/4} I_2(z, \phi))$  is given by

$$\Re(e^{i\pi/4} I_2^{(s)}(z, \phi)) = \frac{\sqrt{2\pi}}{2u^{1/4}} \frac{z^{1/4}}{[z \sin^2 \phi - 1]^{1/4}} [e^{i\pi/4} e^{-\sqrt{u} F_2^>(z, \phi)} + c.c.] ; \quad z > \frac{1}{\sin^2 \phi}, \quad (31)$$

where

$$F_2^>(z, \phi) = \frac{\sin \phi}{2} \left[ \frac{-i}{\sqrt{z} \sin \phi} \left( \ln(\sqrt{z \sin^2 \phi - 1} + \sqrt{z} \sin \phi) + \frac{i\pi}{2} \right) + i\sqrt{z \sin^2 \phi - 1} \right]. \quad (32)$$

The two saddle-point expressions for  $\Re(e^{i\pi/4} I_2(z, \phi))$ , Eqs.(29) and (31), can be represented by a single formula valid for an arbitrary  $z > 0$ :

$$\Re(e^{i\pi/4} I_2^{(s)}(z, \phi)) = \frac{\sqrt{2\pi}}{2u^{1/4}} \Re \left\{ \frac{e^{i\pi/4} e^{-\sqrt{u} F_2^{(+)}(z, \phi)} + e^{-i\pi/4} e^{-\sqrt{u} F_2^{(-)}(z, \phi)}}{[z \sin^2 \phi - 1]^{1/4}} \right\}, \quad (33)$$

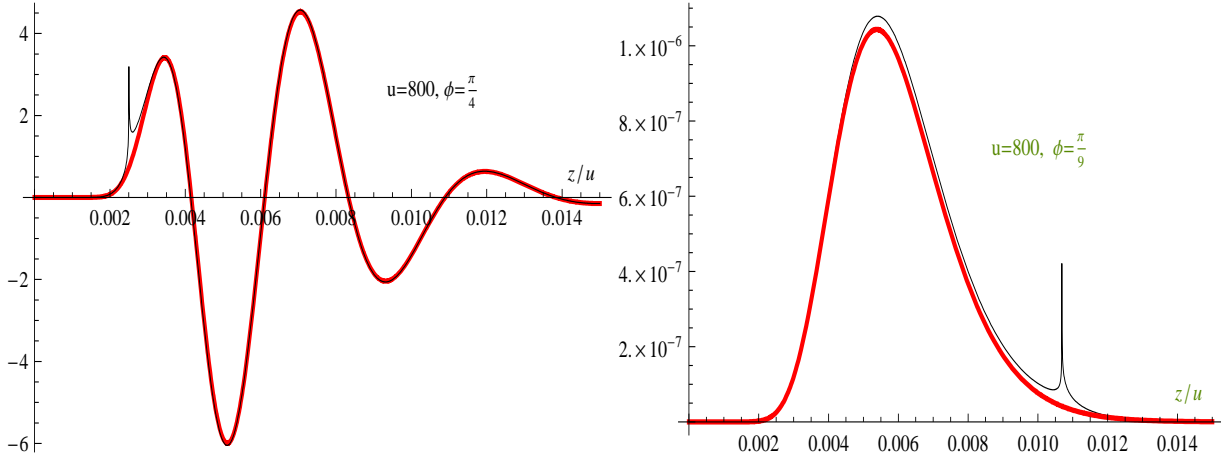
where

$$F_2^{(\pm)}(z, \phi) = \mp i \frac{\sin \phi}{2} \left[ \frac{\ln(\pm i \sqrt{z \sin^2 \phi - 1} \pm i \sqrt{z} \sin \phi)}{\sqrt{z} \sin \phi} - \sqrt{z \sin^2 \phi - 1} \right]. \quad (34)$$

Eqs. (33) and (34) are defined on the complex plane  $z$  with a cut along the ray  $(1/\sin^2 \phi, \infty)$ ; branches of  $(z \sin^2 \phi - 1)^{1/2}$  and  $(z \sin^2 \phi - 1)^{1/4}$  are chosen to be positive on the upper edge of the cut, the (standard) branch of  $\ln w$  is defined by the requirement  $\Im(\ln w) = 0$  at  $w > 0$  and the cut along  $(-\infty, 0)$  on the  $w$ -plane. Accounting for Eqs. (24) and (33), we arrive at the expression for the generating function Eq.(18) in the form ( $u \gg 1$ ):

$$\Phi_{an}^s(u, \phi) = \frac{2\pi}{\Gamma^4(1/4)} \Re \int_C dz \frac{e^{i\pi/4} e^{-\sqrt{u} \mathcal{F}_+(z, \phi)} + e^{-i\pi/4} e^{-\sqrt{u} \mathcal{F}_-(z, \phi)}}{[(z \cos^2 \phi + 1)(z \sin^2 \phi - 1)]^{1/4}}; \quad (35)$$

$$\mathcal{F}_{\pm}(z, \phi) \equiv F_1(z, \phi) + F_2^{(\pm)}(z, \phi). \quad (36)$$



**Figure 3.** The saddle-point integrand (arbitrary units) in Eq.(35) for  $u = 800$  and two different angles  $\phi = \frac{\pi}{4}$  and  $\phi = \frac{\pi}{9}$  (thin solid line) compared with the proper integrand in the exact solution Eq.(14) (thick solid line). The coincidence is very good except in the vicinity of the branch cut point  $z = \sin^{-2} \phi$  where there is an integrable singularity in the saddle-point integrand. As  $u$  increases, this peak singularity moves to the tails of the integrand (to the right tail for  $\phi < \phi_c$  and to the left tail for  $\phi > \phi_c$ ) and thus makes negligible contribution to the  $z$ -integral. An exception is the case of  $\phi \approx \phi_c$  where the peak does not move to the tails. In this case the saddle-point integrand Eq.(35) is no longer valid (see Appendix A).

Here the integration contour  $C = C_0 + C_+^R$  on the complex plane  $z$  with the cuts along the rays  $(-\infty, -1/\cos^2 \phi)$  and  $(1/\sin^2 \phi, \infty)$ , is shown in Fig.5; the chosen branch of  $(z \cos^2 \phi + 1)^{1/4}$  is positive at  $z > -1/\cos^2 \phi$ . The integrand in Eq.(35) is depicted (for different values of  $\phi$ ) in Fig.3 together with the result of the direct numerical evaluation of the integrand (after switching to the  $z$ -variable) in Eq.(14).

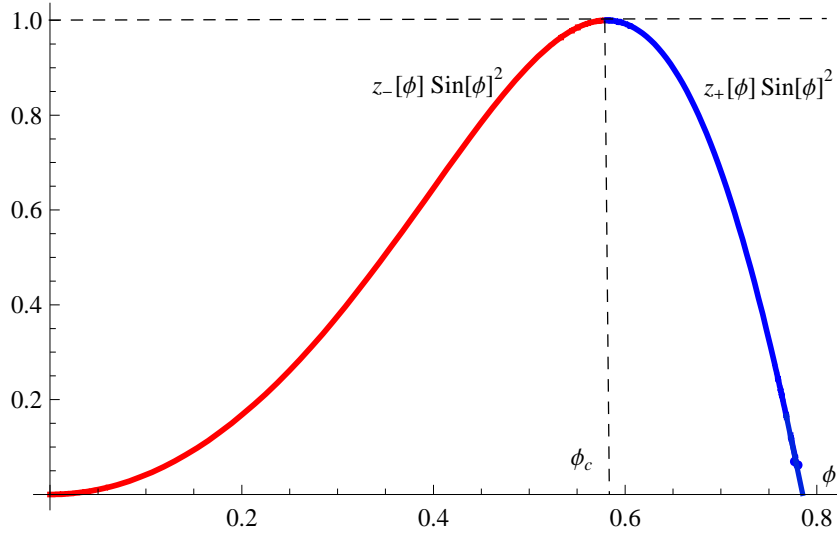
Our next step is the calculation of the integral in Eq.(35).

## 2.2. Saddle-point calculation of the integral in Eq.(35) for $\Phi(u, \phi)$

Saddle-points of the integrand in Eq.(35) are determined by solutions  $z_{\pm}(\phi)$  to the equations

$$\frac{\partial \mathcal{F}_{\pm}(z, \phi)}{\partial z} = \frac{1}{4z} \left[ -\frac{\ln \left( \sqrt{z \cos^2 \phi + 1} + \sqrt{z} \cos \phi \right)}{\sqrt{z}} + \cos \phi \sqrt{z \cos^2 \phi + 1} \right. \\ \left. \pm i \frac{\ln \left( \pm i \sqrt{z \sin^2 \phi - 1} \pm i \sqrt{z} \sin \phi \right)}{\sqrt{z}} \pm i \sin \phi \sqrt{z \sin^2 \phi - 1} \right] = 0. \quad (37)$$

It turns out that the solutions  $z_{\pm}(\phi)$  are real and  $0 \leq z_{\pm}(\phi) \leq 1/\sin^2 \phi$ ; the solution  $z_{-}(\phi)$  exists for  $0 \leq \phi \leq \phi_c$ , while the solution  $z_{+}(\phi)$  exists for  $\phi_c \leq \phi \leq \pi/4$  (we recall that we consider the angle interval  $(0, \pi/4)$ ), see Fig.5. The critical angle  $\phi_c$  is determined by the condition  $z_{\pm}(\phi_c) \sin^2 \phi_c = 1$ , i.e. the solution reaches the origin of



**Figure 4.** Solutions  $z_{\pm}(\phi)$  to the saddle-point equations Eq.(37)

the right cut. With this condition, the equation Eq.(37) results in (for the both signs):

$$Y(\phi_c) = 0, \quad \text{where } Y(\phi) \equiv \ln \left( \frac{\sin \phi}{1 + \cos \phi} \right) + \frac{\cos \phi}{\sin^2 \phi} - \frac{\pi}{2}. \quad (38)$$

This transcendental equation can be represented in a nice form using the parametrization

$$\cot \phi_c = \sinh \left( \frac{x_c}{2} \right), \quad (39)$$

where  $x_c \approx 2.4164...$  is the solution of the equation

$$\sinh x - x = \pi \quad (40)$$

The value of the critical angle  $\phi_c$

$$\phi_c = 0.58060... \quad (41)$$

arises as an important constant also in the calculation of the probability function  $P_{an}(|\psi|^2)$ , section 3.

In the vicinity of this critical angle we have:

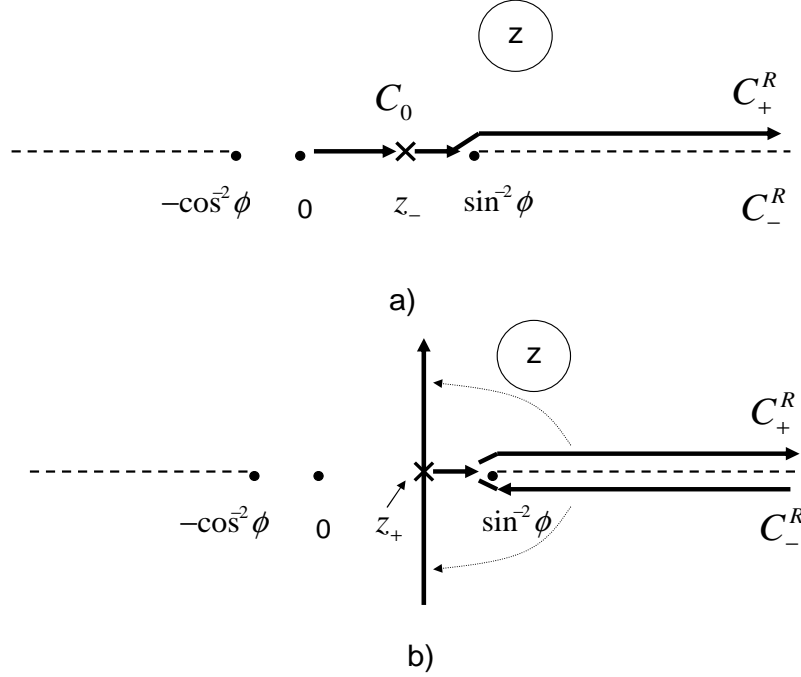
$$\begin{aligned} z_+(\phi) = z_-(\phi) &= \frac{1}{\sin^2(\phi)} \left( 1 - \frac{Y^2(\phi)}{4} \right) \\ &\approx \frac{1}{\sin^2(\phi_c)} - \frac{(\phi - \phi_c)^2}{\sin^4 \phi_c} \left( \frac{1}{\sin^2 \phi_c} - 1 \right), \end{aligned} \quad (42)$$

where  $Y(\phi)$  is given by Eq.(38). At the ends of the angle interval, i.e. at  $\phi = \pi/4$  and  $\phi = 0$ , the solutions to Eq.(37) are given by

$$z_+(\phi) \approx 20 \left( \frac{\pi}{4} - \phi \right) - \frac{1000}{9} \left( \frac{\pi}{4} - \phi \right)^3 \quad (43)$$

$$z_-(0) = \sinh^2(x_0/2) \approx 4.1263...; \quad \text{where } \sinh(x_0) - x_0 = 2\pi. \quad (44)$$

Using Eq.(37), one can represent the saddle-point actions in a following compact form:



**Figure 5.** Contours in the complex plane of  $z$ : a) evaluation of the contribution of the saddle-point  $z_-$ ; b) evaluation of the contribution of the saddle-point  $z_+$ . At the critical angle  $\phi_c$  the saddle points (denoted by a cross) touch the branch cut point  $z = \sin^{-2} \phi$ .

$$\mathcal{F}_{\pm}^s(\phi) = \cos \phi \sqrt{1 + z_{\pm}(\phi) \cos^2 \phi} \pm \sin \phi \sqrt{1 - z_{\pm}(\phi) \sin^2 \phi}. \quad (45)$$

In a similar way, the second derivatives of the actions in the saddle point can be represented as:

$$\left. \frac{\partial^2 \mathcal{F}_{\pm}^s(z, \phi)}{\partial z^2} \right|_{z=z_{\pm}(\phi)} = \frac{1}{4z_{\pm}(\phi)} \left( \frac{\cos^3 \phi}{\sqrt{1 + z_{\pm}(\phi) \cos^2 \phi}} \mp \frac{\sin^3 \phi}{\sqrt{1 - z_{\pm}(\phi) \sin^2 \phi}} \right). \quad (46)$$

It follows from Eq.(46) that the second derivative is positive at  $z_-(\phi)$  ( $\phi < \phi_c$ ) and negative at  $z_+(\phi)$  ( $\phi > \phi_c$ ). Therefore, for  $\phi < \phi_c$ , the contour  $C = C_0 + C_+^R$  (see Fig.5a) goes through the saddle point  $z_-(\phi)$  within the proper Stokes sectors, and the term with  $\mathcal{F}_-$  makes the contribution to the integral in Eq.(35). The term with  $\mathcal{F}_+$  in Eq.(35) does not have a saddle point at  $\phi < \phi_c$  and its contribution is negligible at large  $u$ .

On the contrary, for  $\phi > \phi_c$  the contour  $C$  is not appropriate as it goes within improper Stokes sectors of the saddle point  $z_+(\phi)$ . To overcome this obstacle, let us modify the contour  $C = C_0 + C_+^R$  by adding an additional contour  $C_-^R$  which corresponds

to the lower edge of the cut, see Fig.5b. It is seen easily that this operation does not change the integral Eq.(35) because its integrand is purely imaginary on  $C_-^R$ . Now, the part  $C^R \equiv C_+^R + C_-^R$  of the modified contour can be deformed to the vertical contour which goes through the saddle point  $z_+(\phi)$  within the proper Stokes sectors, Fig.5b. The corresponding saddle-point contribution to the generating function Eq.(35) at  $\phi > \phi_c$  is determined by the  $\mathcal{F}_+(z, \phi)$  term in the integrand. Summarizing these results we arrive at the following asymptotic expression for the generating function:

$$\Phi_{an}^s(u, \phi) = A(\phi) \frac{e^{-\sqrt{u}b(\phi)}}{u^{1/4}} \quad ; \quad u \gg 1. \quad (47)$$

Here the function  $A(\phi)$  outside of a narrow vicinity of the critical angle  $\phi_c$  (see below) is given by

$$A(\phi) = \frac{(2\pi)^{3/2}}{\Gamma^4(1/4)} \frac{2\sqrt{z_{\pm}(\phi)}}{\left| \sin^3 \phi \sqrt{1 + z_{\pm}(\phi) \cos^2 \phi} \mp \cos^3 \phi \sqrt{1 - z_{\pm}(\phi) \sin^2 \phi} \right|^{1/2}}, \quad (48)$$

while the function  $b(\phi)$  is given by Eq.(45):

$$b(\phi) = \mathcal{F}_{\pm}^s(\phi). \quad (49)$$

In these equations the upper (lower) signs stand for  $\phi > \phi_c$  ( $\phi < \phi_c$ ). At the particular angles,  $\phi = \phi_c$ ,  $\phi = \pi/4$  and  $\phi = 0$ , the functions  $b(\phi)$  and  $A(\phi)$  are given by (see expressions Eqs.(43) and (44)):

$$b(\phi = \pi/4) = \sqrt{2} \quad ; \quad b(\phi = 0) = \sqrt{1 + z_-(0)} \approx 2.2641... \quad (50)$$

$$b(\phi = \phi_c + \delta\phi) = \cot \phi_c - \delta\phi + \cot^3 \phi_c \frac{(\delta\phi)^2}{2} + O((\delta\phi)^3).$$

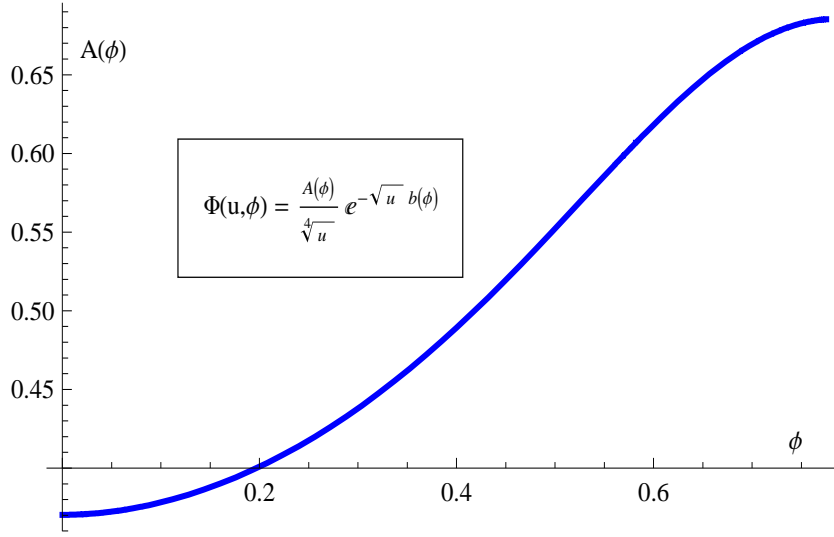
$$A(\phi = \pi/4) = \frac{8 \cdot 2^{1/4} \sqrt{5} \pi^{3/2}}{\Gamma^4(\frac{1}{4})} = 0.6855... , \quad (51)$$

$$A(\phi = 0) = \frac{2(2\pi)^{3/2}}{\Gamma^4(\frac{1}{4})} \sqrt{z_-(0)} \approx 0.3703 \quad (52)$$

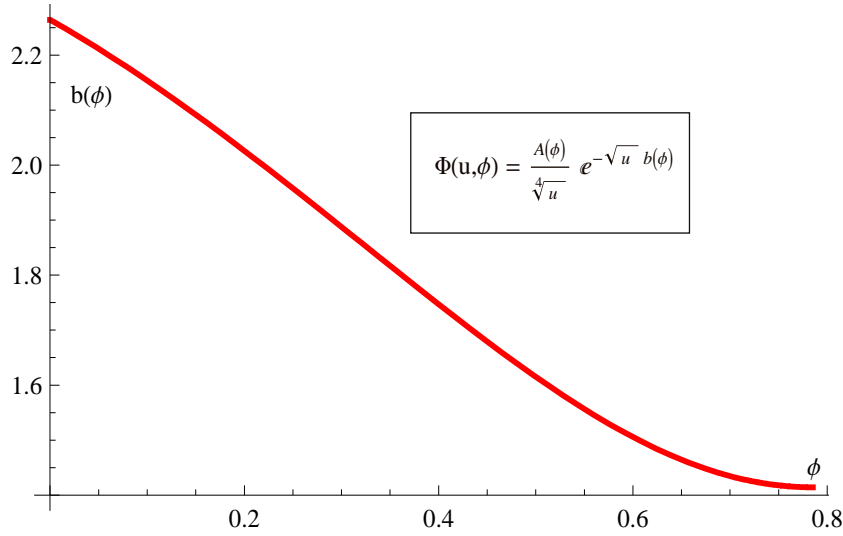
$$A(\phi = \phi_c) = \frac{4 \sqrt{2} \pi^{3/2}}{\Gamma^4(\frac{1}{4}) \sin^2 \phi_c} = 0.6059... . \quad (53)$$

The plots of the functions  $A(\phi)$  and  $b(\phi)$  computed from Eqs.(48) and (49) are given in Fig.6 and Fig.7. Remarkably, the plots which were calculated from different expressions at  $\phi > \phi_c$  and  $\phi < \phi_c$  do not show any singularity at  $\phi = \phi_c$ . The two pieces of the curves match perfectly at the critical angle  $\phi = \phi_c$ .

In the next section we present a different calculation of the function  $b(\phi)$  which does not possess any critical angle by construction and coincides identically with the above saddle-point expressions. As both  $A(\phi)$  and  $b(\phi)$  are expressed through the same solutions  $z_{\pm}(\phi)$  of the saddle-point equation, smoothness of  $b(\phi)$  at  $\phi = \phi_c$  implies also the smoothness of  $A(\phi)$ .



**Figure 6.** The function  $A(\phi)$  in Eq.(47).



**Figure 7.** The function  $b(\phi)$  in Eq.(47).

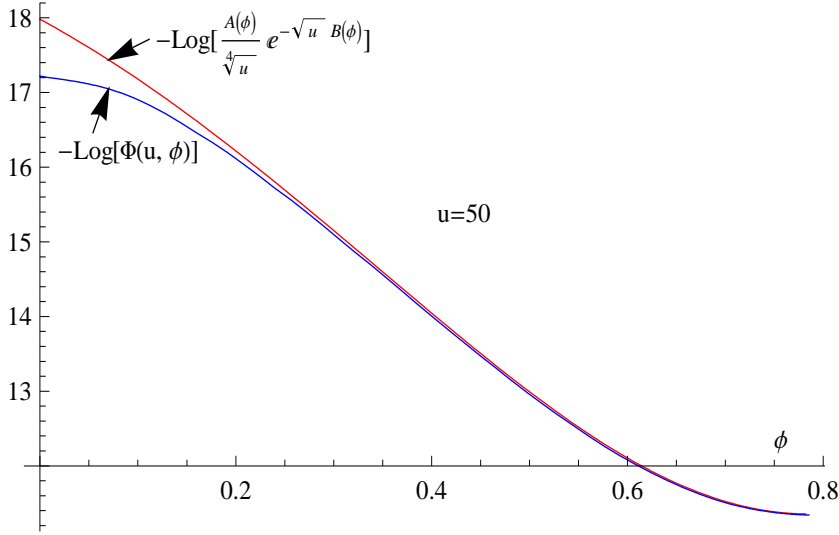
### 2.3. Ordinary differential equation for the exponent $b(\phi)$

Let us look for an asymptotic ( $u \gg 1$ ) solution to the original PDE (10) in the form:  $\Phi(u, \phi) \sim \exp[-u^p b(\phi)]$  where  $m$  and  $b(\phi)$  are to be determined by keeping in the PDE terms of the leading order in  $u$ . We find immediately that  $p = 1/2$  (which is in accordance with (47)) while  $b(\phi)$  obeys the ordinary differential equation (ODE):

$$\frac{3 + \cos 4\phi}{4} \left( \frac{db(\phi)}{d\phi} \right)^2 + \frac{\sin 4\phi}{2} b(\phi) \frac{db(\phi)}{d\phi} + \frac{1 - \cos 4\phi}{4} b^2(\phi) = 1. \quad (54)$$

One can reduce the equation to the form convenient for numerical integration by introducing the function:

$$y(\phi) = \frac{\sqrt{2} b(\phi/2)}{2(1 + \cos^2 \phi)^{\frac{1}{4}}}. \quad (55)$$



**Figure 8.** The comparison of the approximate function  $\Phi(u, \phi)$  given by Eq.(47) and the exact  $\Phi(u, \phi)$  Eq.(14) obtained numerically at  $u = 50$ . Near  $\phi = 0$  the exact function has an extremum to ensure smoothness of the even function  $\Phi(u, \phi) = \Phi(u, -\phi)$ . The approximation is not accurate in the vicinity of  $\phi = 0$ , where the right branch-cut point in Fig.5 moves to infinity. However, it becomes more and more accurate as  $u$  increases. Note that the principle parameter of the approximation  $u^{-1/4} \approx 0.38$  is not very small at  $u = 50$ .

Then Eq.(54) takes the form:

$$\frac{dy}{d\phi} = \pm \frac{\sqrt{1 - y^2 \frac{\sin^2 \phi}{(1 + \cos^2 \phi)^{1/2}}}}{2(1 + \cos^2 \phi)^{3/4}}. \quad (56)$$

The initial conditions for Eqs.(54),(56) follow from Eq.(50):

$$b(\pi/4) = \sqrt{2}, \quad y(\pi/2) = 1. \quad (57)$$

There is an obvious solution to Eq.(54) with the initial condition Eq.(57):

$$b_0(\phi) = \cos \phi + \sin \phi. \quad (58)$$

It corresponds to the choice of sign "+" in Eq.(56). This solution is a growing function of  $\phi$  with the *maximum* at  $\phi = \pi/4$ . Therefore it does not correspond to the saddle-point solution which has a *minimum* at  $\phi = \pi/4$  (see Fig.7 and Eq.(50)). In fact, the solution Eq.(58) corresponds to the particular solution Eq.(11) which we have already discarded on physical grounds. Thus the relevant solution for our problem is the one which corresponds to the sign "minus" in Eq.(56). This ODE can be transformed into the Abel's ODE [26] but it does not belong to the classes with known solutions.

We solved Eq.(56) numerically applying the initial condition Eq.(57) at a point  $\phi = \pi/2 - \delta$  with  $\delta = 10^{-10}$ . We checked that the solution corresponding to the sign "plus" matches the function  $b_0(\phi)$  obtained from Eqs.(55),(58) with the same accuracy. Much less trivial is that the solution for  $b(\phi)$  corresponding to the sign "minus" in Eq.(56) coincides (with the same accuracy) with the saddle-point solution given by

Eqs.(45),(49). Remarkably, the solution to the particular Abel's ODE appeared to be represented in terms of the solution to the transcendental saddle-point equation Eq.(37)! In this connection we would like to remind about another "miracle" of the problem. Namely, the saddle-point solution for  $b(\phi)$  which we obtained from two pieces  $\mathcal{F}_{\pm}^s(\phi)$  expressed through solutions  $z_+(\phi)$  and  $z_-(\phi)$  of the saddle-point equations Eq.(37), appeared to be smooth at the critical angle  $\phi_c$  where the two pieces match perfectly. Now we understand that this is a direct consequence of the fact that  $b(\phi)$  can be obtained from the ODE which has no singularity at  $\phi = \phi_c$ .

This argument is also important to realize that the asymptotic function  $\Phi_{an}(u, \phi)$  (47) is valid also in the vicinity of the critical angle  $\phi_c$  where the saddle-point expression for the integrand in Eq.(35) is no longer valid. As is shown in Appendix A, at  $|\phi - \phi_c| < u^{-1/6}$  the integrand should be modified so that the (fake) singularity at  $z = \sin^{-2} \phi$  is rounded. Then the  $z$ -integral can be computed analytically which results in the same asymptotic Eq.(47) with  $b(\phi)$  given by Eq.(49). We conclude therefore that different procedures for  $|\phi - \phi_c| \gg u^{-1/6}$  and  $|\phi - \phi_c| \ll u^{1/6}$  give the same result. Thus there is no real "critical angle" in the function  $\Phi_{an}(u, \phi)$ , while there is a critical point in the integrand in Eq.(35).

### 3. Probability distribution function $P_{an}(|\psi|^2)$ of anomalously localized eigenstates

The generating function  $\Phi(u, \phi; E)$  allows one to calculate all local statistics of eigenfunctions. The probability distribution function  $P(|\psi|^2)$  is connected with a "joint probability distribution function"  $P(u, \phi)$  (see [18] for details):

$$P(|\psi|^2) = \int_0^\infty du \int_0^\pi d\phi \delta(|\psi|^2 - u \cos^2 \phi) P(u, \phi) = \int_0^\pi \frac{d\phi}{\cos^2 \phi} P\left(\frac{|\psi|^2}{\cos^2 \phi}, \phi\right). \quad (59)$$

The function  $P(u, \phi)$ , in its turn, is related with the generating function  $\Phi(u, \phi)$ . This relation in the limit of a long chain of the length  $L \gg \ell_0$  reads:

$$\begin{aligned} P(u, \phi) &= i \frac{\nu_0(E)}{L\nu(E)u} \partial_u \int_{-i\infty+0}^{+i\infty+0} \frac{dt}{t} e^{4t/\ell_0} \Phi^2(ut, \phi) \\ &= -4i \frac{\nu_0(E)}{L\nu(E)u^2} \int_{-i\infty+0}^{+i\infty+0} dt e^{4t/\ell_0} \Phi^2(ut, \phi), \end{aligned} \quad (60)$$

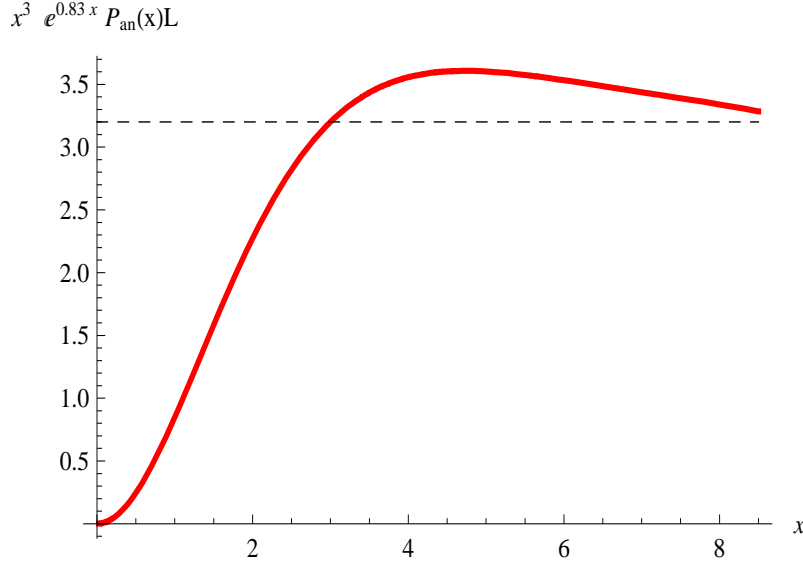
where the localization length  $\ell_0 \gg 1$  (away from the  $E = 0$  anomaly), the averaged DoS  $\nu(E = 0)$ , and the DoS  $\nu_0(E = 0)$  of an ideal (without disorder) chain are given by Eqs.(2) and (3). Our aim is to find the asymptotic form of  $P(|\psi|^2)$  at  $|\psi|^2 \ell_0 \gg 1$ .

The asymptotic form of the function  $P(u, \phi)$  is determined by that of the generating function  $\Phi(u, \phi)$

$$\Phi_{an}^s(u, \phi) = \mathcal{A}(\phi) u^q e^{-\sqrt{u} b(\phi)} \quad ; \quad u \gg 1 \quad (61)$$

represented in the form suitable for both the normal ( $q = 1/4$ ,  $b(\phi) = 2$ ; Eq.(13)) and anomalous ( $q = -1/4$ ; Eq.(47)) functions.





**Figure 9.** The ratio of the probability distribution  $P(x)$  (computed numerically from the exact solution Eq.(14)) to the asymptotic expression Eq.(67) obtained for  $x = |\psi|^2 \ell_0 \gg 1$ . The depletion at  $x < 4$  arises because of the  $1/x$  behavior of the exact distribution at small  $x$  as compared to  $1/x^3$  behavior of the asymptotic expression. The quasi-constant behavior at  $x > 4$  (with the value of the constant close to that of Eq.(68) depicted by the dashed line) indicates on the setting up of the pre-exponent  $\propto 1/x^3$ .

Plugging this function into Eq.(60) and doing the saddle-point integration over  $t$  one obtains:

$$P(u, \phi) = 2\sqrt{\pi} \mathcal{A}^2(\phi) \frac{\nu_0(E=0)}{L \nu(E=0)} \left(\frac{\ell_0}{4}\right)^{4q+1/2} [b(\phi)]^{4q+1} u^{4q-3/2} e^{-u \ell_0 b^2(\phi)/4}. \quad (62)$$

Now using the  $\pi/2$ -periodicity of the integrand in Eq.(59) one finally arrives at:

$$P(|\psi|^2) = C (|\psi|^2 \ell_0)^{4q-3/2} \int_0^{\pi/2} \frac{\mathcal{A}^2(\phi)}{[\cos \phi]^{8q-1}} [b(\phi)]^{4q+1} e^{-\frac{|\psi|^2 \ell_0}{4} \frac{b^2(\phi)}{\cos^2 \phi}} d\phi, \quad (63)$$

where

$$C = \frac{2\sqrt{\pi}}{4^{4q}} \frac{\ell_0^2}{L} \frac{\nu_0(E=0)}{\nu(E=0)}. \quad (64)$$

In the limit  $|\psi|^2 \ell_0 \gg 1$ , the major contribution to Eq.(63) comes from the vicinity of the minimum of the function

$$B(\phi) \equiv \left(\frac{b(\phi)}{\cos \phi}\right)^2 \quad (65)$$

entering the exponent. Outside the anomaly (or for the continuum model) the function  $b(\phi) = 2$  (see Eq.(13), so the minimum value of  $B(\phi)$  is achieved at  $\phi = 0$ . Performing the saddle-point integration in Eq.(63) we obtain the announced expression Eq.(9) for the asymptotic of the “normal” probability distribution function  $P_{norm}(|\psi|^2)$  of eigenstates. It is interesting that this asymptotic form coincides with the exact function  $P_{norm}(|\psi|^2)$  [18] for any value of  $|\psi|^2$ .

For the center-of-band anomaly, the function  $b(\phi)$  is more complicated. It follows from Eq.(50) that the function  $B(\phi)$  has its minimum exactly at the critical angle  $\phi_c$  (Eqs.(38)-(41)):

$$B(\phi_c + \delta\phi) = \frac{1}{\sin^2(\phi_c)} + \frac{(\delta\phi)^2}{\sin^4(\phi_c)} + \dots \quad (66)$$

Thus the critical angle remarkably appears again in the theory even though we have shown that the function  $\Phi_{an}(u, \phi)$  is smooth at  $\phi = \phi_c$ .

The saddle-point integration in Eq.(63) leads to the following result:

$$P_{an}(|\psi|^2) = C(\phi_c) \frac{1}{\ell_0 L |\psi|^6} \exp\left(-\frac{|\psi|^2 \ell_0}{4 \sin^2(\phi_c)}\right), \quad (67)$$

where

$$C(\phi_c) = \frac{64\pi\sqrt{2}}{\Gamma^4\left(\frac{1}{4}\right)} \frac{\cos^3 \phi_c}{\sin^2 \phi_c} \approx 3.20. \quad (68)$$

and

$$\frac{1}{4 \sin^2(\phi_c)} = 0.8310... < 1, \quad (69)$$

Equation (67) is the main result of our paper.

#### 4. $P(|\psi|^2)$ at small $|\psi|^2 \ell_0 \ll 1$ .

In this section we consider the behavior of the eigenfunction amplitude distribution function  $P(|\psi|^2)$  at small values of  $|\psi|^2 \ell_0$ . Generically, the small amplitudes  $|\psi|^2 \ell_0 \ll 1$  arise either (i) due to localization when the observation point  $\mathbf{r}$  in  $\psi = \psi(\mathbf{r})$  lies outside the localization volume, or (ii) due to the proximity of the observation point to the node of the wave function. In the case (i) the amplitude of exponentially localized eigenfunction cannot be smaller than  $|\psi| \sim \ell_0^{-1/2} e^{-L/2\ell^{\text{ext}}}$ , while in the case (ii) the amplitude  $|\psi|$  can be arbitrary small. It is clear that the case (i) is realized with almost certainty in a large sample, while the case (ii) has small probability proportional to the small distance of the observation point from the node. This should lead to the drastically different behavior of the distribution function for  $|\psi|^2 \ell_0 \gg e^{-L/\ell^{\text{ext}}}$  (case (i)) and for  $|\psi|^2 \ell_0 \ll e^{-L/\ell^{\text{ext}}}$  (case (ii)). Our approach based on the exact solution of the *stationary* (with respect to the coordinate along the chain) evolution equation (10) is capable of describing only the case (i), as the crossover to the alternative case (ii) and the corresponding solution for the generating function are essentially  $L$ -dependent.

Furthermore, one can argue that for the case of pure exponential localization the asymptotic behavior of  $P(|\psi|^2)$  at small  $|\psi|^2 \ell_0$  ( $e^{-L/\ell^{\text{ext}}} \ll |\psi|^2 \ell_0 \ll 1$ ) should be always  $P(|\psi|^2) = C_{\text{norm}}/|\psi|^2$ . Indeed, in this case the normalization integral is logarithmically divergent and dominated by  $|\psi|^2 \ell_0 \sim e^{-L/\ell^{\text{ext}}}$ . Thus the normalization constant  $C_{\text{norm}} \propto 1/L$ , as it should be in order to make the first moment  $\langle |\psi|^2 \rangle = \frac{1}{L}$  as the eigenfunction normalization requires. Should the profile of the localization tail

be of the form  $L^{-\alpha} e^{-L/\ell^{\text{ext}}}$  with an extra power-law pre-exponent, the characteristic *sub-leading* terms appear in  $P(|\psi|^2)$ :

$$P(|\psi|^2) = \frac{C_{\text{norm}}}{|\psi|^2} \left(1 - \frac{\alpha}{\ln |\psi|^2} + \dots\right), \quad 1 \gg |\psi|^2 \ell_0 \gg e^{L/\ell^{\text{ext}}}. \quad (70)$$

Thus studying details of the distribution function  $P(|\psi|)$  at small amplitudes one may infer information about the profile of the tail of the wave function.

In this section we briefly discuss how the principle term in Eq.(70) arises in our formalism. To begin with we note that according to Eq.(59), the term  $|\psi|^{-2}$  at  $|\psi| \ll 1$  may arise only when  $P(u \ll 1, \phi) \propto u^{-1}$ . This means that the integral in Eq.(60)

$$\int_{-i\infty+0}^{+i\infty+0} \frac{dt}{t} e^{4t/\ell_0} \Phi^2(ut, \phi) \quad (71)$$

must be proportional to  $u$  at  $u \ll 1$ . Then one immediately concludes that the function  $\Phi(u, \phi)$  should have a singularity at  $u = 0$ . Indeed, in case of a regular  $u$ -expansion, the term  $\propto u$  in  $\Phi^2(u, \phi)$  would result in a linear in  $u$  contribution in Eq.(71) proportional to the integral:

$$\int_{-i\infty+0}^{+i\infty+0} dt e^{4t/\ell_0} = 0.$$

To obtain the desired dependence  $P(u \ll 1, \phi) \propto u^{-1}$ , one has to assume that there is a term  $\propto u \ln u$  in the expansion of  $\Phi(u, \phi)$ . Then the corresponding integral in Eq.(71)

$$\int_{-i\infty+0}^{+i\infty+0} dt \ln t e^{4t/\ell_0} = -2\pi i \int_0^\infty dt e^{-4t/\ell_0} = -2\pi i \frac{\ell_0}{4}. \quad (72)$$

would be non-zero and result in  $P(u, \phi) \propto u^{-1}$ . We see, therefore, that the term  $u \ln u$  in the expansion of  $\Phi(u, \phi)$  at small  $u$  is the direct consequence of exponential localization.

Next, one can check that that the “normal” (away from the anomaly) generation function Eq.(13) has, indeed, the expansion with  $\ln u$ -terms:

$$\Phi_{\text{norm}}(u, \phi) = \frac{1}{\pi} \sum_{n=0}^{\infty} (g_n + f_n \ln u) u^n, \quad (73)$$

where the first few coefficients of expansion are given by:

$$\begin{aligned} g_0 &= 1, \quad f_0 = 0, \quad f_1 = 1, \quad g_1 = 2\gamma - 1, \\ f_2 &= \frac{1}{2}, \quad g_2 = \gamma - \frac{5}{4}, \end{aligned} \quad (74)$$

where  $\gamma = 0.577216\dots$  is the Euler constant.

A natural *assumption* would be that the generating function at the anomaly  $\Phi_{\text{an}}(u, \phi)$  has the same type of expansion Eq.(73) but with the  $\phi$ -dependent coefficients  $g_n(\phi)$  and  $f_n(\phi)$ :

$$\Phi_{\text{an}}(u, \phi) \stackrel{?}{=} \sum_{n=0}^{\infty} [g_n(\phi) + f_n(\phi) \ln u] u^n. \quad (75)$$

If so, one can find the coefficients by plugging the series Eq.(75) directly into Eq.(10). Then one obtains the chain of recursive equations:

$$\left[ \frac{3 + \cos 4\phi}{4} \partial_\phi^2 + \left( n - \frac{3}{2} \right) \sin 4\phi \partial_\phi + n(n-1) - (n-1)(n-2) \cos 4\phi \right] g_n(\phi) +$$

$$+ [\sin 4\phi \partial_\phi + 2n - 1 + (3 - 2n) \cos 4\phi] f_n(\phi) = g_{n-1}(\phi) \quad (76)$$

$$\left[ \frac{3 + \cos 4\phi}{4} \partial_\phi^2 + \left( n - \frac{3}{2} \right) \sin 4\phi \partial_\phi + n(n-1) - (n-1)(n-2) \cos 4\phi \right] f_n(\phi) = f_{n-1}(\phi). \quad (77)$$

For  $n = 1$  the equation (77) takes the form

$$\left[ \frac{3 + \cos 4\phi}{2} \partial_\phi - \sin 4\phi \right] \partial_\phi f_1(\phi) = 0, \quad (78)$$

which determines the derivative of  $f_1(\phi)$ :

$$\partial_\phi f_1(\phi) = \frac{c_f}{\sqrt{3 + \cos(4\phi)}}. \quad (79)$$

Now we are going to apply the condition of periodicity of  $\Phi(u, \phi)$  as the function of the angle  $\phi$ . Since the left-hand side of this equation is a derivative of a periodic function, its integral over the period must vanish. This can be provided only with the choice  $c_f = 0$ . Hence, the function  $f_1(\phi)$  is a constant:

$$f_1(\phi) = F_1, \quad (80)$$

where the constant  $F_1$  cannot be fixed by the homogeneous equation (78).

However, it appears that the requirement of periodicity of the function  $g_1(\phi)$  helps to fix the constant  $F_1$ . Indeed, the equation for this function is:

$$\left[ \frac{3 + \cos 4\phi}{4} \partial_\phi - \frac{1}{2} \sin 4\phi \right] \partial_\phi g_1(\phi) = g_0(\phi) - (1 + \cos 4\phi) F_1, \quad (81)$$

where  $g_0(\phi) \equiv \mathcal{P}_{an}(\phi)$  is given by Eq.(15). Looking for the solution in the form  $\partial_\phi g_1(\phi) = c_g(\phi) g_0(\phi)$  and taking into account that  $g_0(\phi)$  obeys the homogeneous equation (for the zero R.H.S.), we obtain the following equation for  $c_g(\phi)$ :

$$\partial_\phi c_g(\phi) = \frac{4}{3 + \cos 4\phi} - \frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}} \frac{1 + \cos(4\phi)}{\sqrt{3 + \cos(4\phi)}} F_1. \quad (82)$$

Using once again the periodicity condition, we must require the integral over the period of each side of the above equation to vanish. This determines uniquely the value of  $F_1$ :

$$F_1 = \frac{\sqrt{2}}{4}. \quad (83)$$

We see that the solution can be found uniquely only if one assumes the periodicity (and hence smoothness) of the function  $\Phi(u, \phi)$ , and this solution corresponds to  $f_1(\phi) \neq 0$ . This means that it would not be possible to find any periodic solution without a term  $\propto u \ln u$  in the series Eq.(75) for  $\Phi_{an}(u, \phi)$ . Thus the periodicity requires the singular expansion at  $u = 0$  with certainty. We note that the assumption of smoothness was the key point to obtain the exact solution Eq.(14) [17, 18].

With the coefficient  $f_1(\phi) = F_1$  established one immediately finds the leading term in the  $P(|\psi|^2)$  at small  $|\psi|^2$ :

$$P(|\psi|^2) = \frac{\Gamma^4(1/4)}{16\pi^2} \frac{\ell_0}{L} \frac{1}{|\psi|^2} = \frac{\ell^{\text{ext}}}{L} \frac{1}{|\psi|^2}, \quad (e^{-L/\ell^{\text{ext}}} \ll |\psi|^2 \ell_0 \ll 1). \quad (84)$$

As was expected, the numerical coefficient in Eq.(84) exactly corresponds to the replacement  $\ell_0 \rightarrow \ell^{ext}$  in the leading term of expansion of Eq.(9) in accordance with Eq.(3).

However, finding sub-leading terms in  $P(|\psi|^2 \rightarrow 0)$  is a separate non-trivial problem. We leave its complete study for future publications, outlining here only the origin of the difficulties. The point is that one can obtain the *formal* series of the type Eq.(75) for  $\Phi_{an}(u, \phi)$  with the coefficients  $g_n(\phi)$  and  $f_n(\phi)$  represented by a two-fold integrals. To this end we exploit again the integral representation Eq.(16) of the Whittaker functions. Plugging it into the exact solution Eq.(14) we do the  $\lambda$ -integration exactly using the well-known integral [25]:

$$\int_0^\infty \frac{d\lambda}{\lambda^2} \exp^{-\frac{a}{\lambda} - b\lambda} = 2 \left( \frac{b}{a} \right)^{\frac{1}{2}} K_1(2\sqrt{ab}). \quad (85)$$

The result is expressed through the two-fold integral:

$$\begin{aligned} \Phi_{an}(u, \phi) &= \frac{2^{5/2}}{\Gamma^4(1/4)} \sqrt{u} \operatorname{Re} \int_1^\infty \frac{dt_1}{(t_1^2 - 1)^{\frac{3}{4}}} \int_1^\infty \frac{dt_2}{(t_2^2 - 1)^{\frac{3}{4}}} \\ &K_1 \left( \sqrt{u} \sqrt{\frac{\epsilon}{2} \ln \left( \frac{t_1 + 1}{t_1 - 1} \right) + \frac{\bar{\epsilon}}{2} \ln \left( \frac{t_2 + 1}{t_2 - 1} \right)} \sqrt{\bar{\epsilon} t_1 \cos^2 \phi + \epsilon t_2 \sin^2 \phi} \right) \\ &\times \left[ \frac{\epsilon \ln \left( \frac{t_1 + 1}{t_1 - 1} \right) + \bar{\epsilon} \ln \left( \frac{t_2 + 1}{t_2 - 1} \right)}{\bar{\epsilon} t_1 \cos^2 \phi + \epsilon t_2 \sin^2 \phi} \right]^{1/2}, \end{aligned} \quad (86)$$

where  $\epsilon = e^{i\pi/4}$ ,  $\bar{\epsilon} = e^{-i\pi/4}$ .

In Eq.(86) one can immediately recognize the combination  $\sqrt{u} K_1(\sqrt{u} \dots)$  which enters Eq.(13) and which generates the series Eq.(75). The coefficients  $g_n(\phi) = g_n^{(1)}(\phi) + g_n^{(2)}(\phi)$  and  $f_n(\phi) = g_n^{(1)}(\phi) (f_n/g_n)$  in the corresponding series for  $\Phi_{an}(u, \phi)$  are expressed in terms of the coefficients  $g_n$  and  $f_n$  appearing in the expansion Eq.(73) of  $2\sqrt{u} K_1(2\sqrt{u})$  and the two-fold integrals:

$$\begin{aligned} g_n^{(1)}(\phi) &= c_n g_n \operatorname{Re} \int_1^\infty \frac{dt_1}{(t_1^2 - 1)^{\frac{3}{4}}} \int_1^\infty \frac{dt_2}{(t_2^2 - 1)^{\frac{3}{4}}} \\ &\times [G(t_1, t_2)]^n [T_\phi(t_1, t_2)]^{n-1}, \end{aligned} \quad (87)$$

$$\begin{aligned} g_n^{(2)}(\phi) &= c_n f_n \operatorname{Re} \int_1^\infty \frac{dt_1}{(t_1^2 - 1)^{\frac{3}{4}}} \int_1^\infty \frac{dt_2}{(t_2^2 - 1)^{\frac{3}{4}}} \\ &\times L_\phi(t_1, t_2) [G(t_1, t_2)]^n [T_\phi(t_1, t_2)]^{n-1} \end{aligned} \quad (88)$$

of the three functions:

$$G(t_1, t_2) = \epsilon \ln \left( \frac{t_1 + 1}{t_1 - 1} \right) + \bar{\epsilon} \ln \left( \frac{t_2 + 1}{t_2 - 1} \right), \quad (89)$$

$$T_\phi(t_1, t_2) = \bar{\epsilon} t_1 \cos^2 \phi + \epsilon t_2 \sin^2 \phi \quad (90)$$

$$L_\phi(t_1, t_2) = \ln [G(t_1, t_2) T_\phi(t_1, t_2)/8], \quad (91)$$

where  $c_n = \frac{2^{3(1-n)}}{\Gamma^4(1/4)}$ .

One can check that  $g_0 \equiv g_0^{(1)}$  coincides with the phase distribution function  $\mathcal{P}_{an}(\phi)$  defined in Eq.(15). Furthermore,  $f_1(\phi)$  appears to be manifestly  $\phi$ -independent and

coincides with  $F_1$  found above (see Eq.(83)). The functions  $g_1^{(1)}(\phi)$  (which is also  $\phi$ -independent) and  $g_1^{(2)}(\phi)$  are also well defined.

However, starting from  $n = 2$  there is a problem in Eq.(87). As the function  $T_\phi(t_1, t_2)$  grows linearly with  $t_{1,2}$ , the integrals in Eq.(87) are divergent for all  $n \geq 2$ . This signals that the expansion Eq.(75) for  $\Phi_{an}(u, \phi)$  breaks down. The reason is that Eq.(75) does not guarantee the correct, decaying at large  $u$  behavior of the generating function  $\Phi(u, \phi)$ . Thus an *additional series* in  $\Phi_{an}(u, \phi)$  may be required to cancel possible divergence at  $u \rightarrow \infty$  of the function obtained by the analytical continuation of the series Eq.(75). As the result the sub-leading term in the expansion of  $\Phi_{an}(u, \phi)$  at small  $u$  is not proportional to  $u^2 \ln u$  (as for the generating functions away from the  $E = 0$  anomaly) but could be much larger. This anomaly deserves a separate investigation.

## 5. Discussion and Conclusion

The goal of this paper was two-fold. The first objective was an asymptotic analysis of the exact solution Eq.(14) for the anomalous (at the center-of-band anomaly,  $E = 0$ ) generating function  $\Phi_{an}(u, \phi)$  at large values of  $u \gg 1$ . The corresponding result is expressed by Eqs.(45) and (47)-(49).

Knowing this asymptotic one can compute various quantities of interest related with the local statistics of eigenfunction amplitudes. The simplest one is the distribution function of the eigenfunction amplitudes  $P(|\psi|^2)$  which behavior at large  $|\psi|^2$  gives an idea about the probability of anomalously strongly localized states. To find this asymptotic form at the center-of-band anomaly was our principal physical objective. We managed to obtain the asymptotic expression for  $P_{an}(|\psi|^2)$  in a compact form Eqs.(67) and (68). The result is a bit surprising, as it shows a re-entrant behavior summarized in Fig.1, which points out on the two competing physical phenomena behind it. Another indication of the same phenomena was first found in our earlier works [17, 18] where we noticed two different scales characterizing the moments of  $|\psi|^2$ .

We also analyzed the asymptotic of  $\Phi_{an}(u, \phi)$  at small values of  $u$ . The leading term  $\Phi_{an}(0, \phi)$  gives the distribution function of phases  $\mathcal{P}(\phi)$  [8, 9] which is related with the distribution of scattering phases. The next-to leading term  $\propto u \ln u$  contains information about the tail of the typical wave function. We have computed this term and shown that it is compatible with the exponential localization with the Lyapunov exponent found in Refs. [8, 9]. We have also shown how the sub-leading term  $u \ln u$  results in the universal leading behavior of  $P(|\psi|) \propto |\psi|^{-2}$  at small  $|\psi|^2$ .

However, it appears that computing the further terms of expansion at small  $u$  (which contain information about the pre-exponential behavior of the tail of localized eigenfunctions) is a non-trivial problem which deserves further investigation.

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### **Appendix A. Generating function $\Phi_{an}(u, \phi)$ in the vicinity of the critical angle $\phi_c$**

The saddle-point derivation of the large  $u$  asymptotic of the anomalous generating function  $\Phi_{an}(u, \phi)$  in the form (47) with the angular dependent pre-exponential function (6) is valid everywhere except for a close vicinity of the critical angle  $\phi_c$ , where the saddle-point solutions  $z_{\pm}(\phi)$  to Eq.(37) are close to the branching point  $1/\sin^2 \phi$  so that the power expansion of the actions  $\mathcal{F}_{\pm}(z, \phi)$  breaks down.

To find the width of the critical region of small  $\delta\phi = \phi - \phi_c$ , we introduce the distances  $\Delta_{\pm}$  between the saddle-points  $z_{\pm}(\phi)$  and the right branching point  $z = 1/\sin^2 \phi$  :

$$\Delta_{\pm} \equiv \frac{1}{\sin^2 \phi} - z_{\pm}(\phi) \propto (\delta\phi)^2. \quad (\text{A.1})$$

The latter estimate follows from Eq.(42) in the vicinity of the critical angle; the dependence  $\Delta_{\pm} \sim (\delta\phi)^2$  is clearly seen in Fig.4.

The previous saddle-point approach to Eq.(35) is justified as long as the width of the saddle-point peaks

$$|z - z_{\pm}(\phi)| \sim u^{-1/4} |\partial^2 \mathcal{F}_{\pm}(z, \phi) / \partial z^2|^{-1/2} \sim u^{-1/4} (\Delta_{\pm})^{1/4}$$

is much smaller than the distance  $\Delta z_{\pm}$  from the right branching point. It follows from here that  $|\delta\phi|$  should be greater than  $u^{-1/6}$ .

In the narrow region (of width  $|\delta\phi| \leq u^{-1/6}$ ) around the critical angle  $\phi_c$ , the regular series expansion of the actions  $\mathcal{F}_{\pm}(z, \phi)$  breaks down and the previous saddle-point approach is not applicable. To treat this narrow critical region, we will develop a modified approach. In fact, we will restrict the analysis to even narrower vicinity of  $\phi_c$ :  $|\delta\phi| \ll u^{-1/6}$ , which is sufficient for the calculation of the eigenstates distribution function  $P_{an}(|\psi|^2)$  performed in the section 3.

Note that the integrand in Eq.(35) needs revision in the critical domain, too. This is because the saddle point estimate Eq.(33) for the integral  $I_2(z, \phi)$  Eq.(19) does not work when the two saddle points  $t_{\pm}$  Eq.(25) approach each other (both go to zero at  $z \rightarrow 1/\sin^2 \phi$ ). The saddle-point estimate of the integral is valid only as long as the width of the saddle-point regions  $|t - t_{\pm}| \sim u^{-1/4} |\delta z|^{-1/4} / \sin \phi$  is small as compared to the distance  $|t_+ - t_-| \sim |\delta z|^{1/2} \sin \phi$  between the two saddle points. Here we represented the integration variable  $z$  in the form:

$$z = \frac{1}{\sin^2 \phi} + \delta z. \quad (\text{A.2})$$

From the above estimates one finds that the saddle-point approach is not applicable when  $|\delta z| \leq u^{-1/3}$  and therefore  $|t_{\pm}| \leq u^{-1/6}$ . For small  $|\delta z| \ll 1$  the leading contribution to the integral  $I_2[C] = I_2[C_+]$  (see Eq.(28) and the discussion around it) comes from a narrow vicinity of the origin  $t = 0$ . Expanding the function  $f_2(t, z, \phi)$  Eq.(21) in small  $t$  (and keeping only linear terms in  $\delta z$ ), we arrive at the following expression for  $I_2[C]$ :

$$I_2[C] = e^{-\frac{3\pi}{4}i} e^{-\frac{\pi\sqrt{u}\sin\phi}{4}} \int_{-\infty}^{\infty} dt e^{i\frac{\sqrt{u}\sin\phi}{2}\left[\frac{t^3}{3} - t\delta z \sin^2\phi\right]}. \quad (\text{A.3})$$

This expression and the relation Eq.(27) determine the quantity of our interest,  $\Re[e^{i\pi/4}I_2(z, \phi)]$ ; the integral representation (18) for the anomalous generating function  $\Phi_{an}(u, \phi)$  takes the form:

$$\Phi_{an}(u, \phi) = \frac{2^{1/3}(2\pi)^{3/2} \sin^{2/3}(\phi) u^{1/12}}{\Gamma^4(1/4)} e^{-\sqrt{u}[\cot(\phi) - \sin(\phi) Y(\phi)/2]} \int_{-\infty}^{\infty} d(\delta z) e^{-\sqrt{u} \delta z \sin^3(\phi) Y(\phi)/4} \text{Ai}\left(-2^{-2/3} \sin^{8/3}(\phi) u^{1/3} \delta z\right), \quad (\text{A.4})$$

with  $\text{Ai}(x)$  being the Airy function. According to Eq.(38) the function  $Y(\phi) \sim \delta\phi$  in the critical region, so the linear in  $\delta z$  term in the exponent of the integrand can be safely omitted. Indeed, due to the convergence of the integral of the Airy function, an estimate for typical  $\delta z$  in the integral Eq.(A.4) is  $\delta z \sim u^{-1/3}$ , hence the linear in  $\delta z$  term in the exponent is estimated as  $u^{1/2}\delta\phi\delta z \sim u^{1/6}\delta\phi$  which is negligible in the considered critical region  $|\delta\phi| \ll u^{-1/6}$ . Neglecting the linear in  $\delta z$  term in the exponent and calculating the remaining integral of Airy function, we obtain  $\Phi_{an}(u, \phi)$  in the vicinity of the critical angle  $\phi_c$  in the form Eq.(47) with the pre-exponential function  $A(\phi) \rightarrow A_c(\phi)$ :

$$A_c(\phi) = \frac{2(2\pi)^{3/2}}{\Gamma^4(1/4) \sin^2\phi_c}. \quad (\text{A.5})$$

This expression matches perfectly the out-of-critical expression Eq.(6) when the latter is formally extended to the critical region  $\phi \approx \phi_c$ , where  $z_{\pm}(\phi) \approx 1/\sin^2\phi$ . This comparison completes our derivation of the asymptotic of the generating function  $\Phi_{an}(u, \phi)$  both in and out of the “critical region” and shows that  $\Phi_{an}(u, \phi)$  is a smooth function of  $\phi$  even in the vicinity of the “critical angle”  $\phi_c$ .

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